March 2, 1865.

Major-General SABINE, President, in the Chair.

In accordance with the Statutes, the names of the Candidates for election into the Society were read, as follows:—

James Abernethy, Esq., C.E.
A. Leith Adams, M.B.
Alexander Armstrong, M.D.
William Baird, M.D.
George Bishop, Esq.
John Charles Bucknill, M.D.
Lieut.-Col. Cameron, R.E.
Henry Christy, Esq.
The Hon. James Cockle.
The Rev. William Rutter Dawes.
W. Boyd Dawkins, Esq.
Henry Dircks, Esq.
Thomas Rowe Edmonds, Esq.
Professor Henry Fawcett.
Peter Le Neve Foster, Esq.
Sir Charles Fox, C.E.
Archibald Geikie, Esq.
George Gore, Esq.
Professor Robert Grant.
George Robert Gray, Esq.
William Augustus Guy, M.B.
Capt. Robert Wolseley Haig, R.A.
George Harley, M.D.
Benjamin Hobson, M.B.
William Huggins, Esq.
Fleeming Jenkin, Esq., C.E.
Edmund C. Johnson, M.D.
Henry Letheby, M.B.
Professor Leone Levi.
Waller Augustus Lewis, M.B.
John Robinson M'Clean, Esq., C.E.
Capt. Sir F. Leopold M'Clintock, R.N.
Robert M'Donnell, M.D.
Hugo Müller, Esq., Ph.D.
Charles Murchison, M.D.
Andrew Noble, Esq., C.E.
Sir Joseph P. Olliffe, M.D.
William Kitchen Parker, Esq.
William Henry Perkin, Esq.
Thomas Lamb Phipson, Esq., Ph.D.
Charles Bland Radcliffe, M.D.
Lovell Reeve, Esq.
John Russell Reynolds, M.D.
Thomas Richardson, Esq., Ph.D.
Wm. Henry Leighton Russell, Esq.
Edward Henry Sieveking, M.D.
Alfred Tennyson, Esq., D.C.L.
George Henry Kendrick Thwaites, Esq.
The Rev. Henry Baker Tristram.
Lieut.-Col. James Thomas Walker, R.E.
A. T. Houghton Waters, M.D.
Charles Wye Williams, Esq.
Henry Worms, Esq.

The following communications were read:—

I. "On the Quadric Inversion of Plane Curves." By T. A. Hirst, F.R.S. Received February 16, 1865.

Introduction.

1. The method of inversion which forms the subject of the present paper is an immediate generalization of that now universally employed. In place of a fixed circle with the origin at its centre, any fixed conic (*quadric*)

whatever is taken, as a fundamental curve, and the origin is placed anywhere in its plane. In this manner many descriptive relations which in the ordinary theory are masked, regain the generality and prominence to which they are entitled. Having long ago convinced myself of the *utility* of this generalized method of inversion, I deem it desirable to establish, for the sake of future reference, its chief general principles. With the view of securing the greatest possible familiarity with the effects of inversion, I employ purely geometrical considerations, and everywhere give preference to a direct and immediate contemplation of the several geometrical forms which present themselves. The examples occasionally introduced, are given for the sake of illustration merely; they do not exhibit the full power of the method. Moreover, to prevent, as much as possible, the extension of a paper intended for publication in the *Proceedings of the Royal Society,* no attempt has been made to subject such special cases to exhaustive treatment. The figures are, for the most part, simple; the fundamental one being given, the rest may readily be drawn or imagined; when treating of the effects of inversion on the higher singularities of curves, however, I have thought it desirable to refer by the initials (𝔄. 𝔈.) to articles and figures in Plücker's elaborate work on the *Theorie der Algebraischen Curven.* The relation which the present method bears to the still more general one of *quadric transformation,* as developed in 1832 by Steiner in his *Geometrische Gestalten,* and by Magnus in the eighth volume of Crelle's *Journal,* offers several points of interest to which I propose to return on a future occasion*.

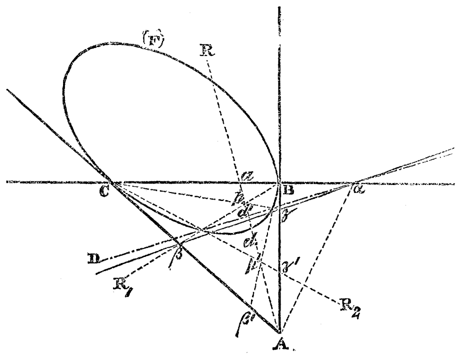
Definitions.

2. Two points, p and p', conjugate to each other with respect to a fixed *fundamental conic* (F), and likewise collinear with any fixed *origin* A in the plane of the latter, are said to be inverse to each other, relative to that conic and origin. In other words, the inverse of a point is the intersection of its polar, relative to (F), and the line which connects it with the origin A. From this the following property is at once deduced:

i. *The several pairs of inverse points* p, p′, *on any line* R *through the origin* A, *form an involution, the foci of which are the intersections, real or imaginary, of that line and the fundamental conic* (F).

* I have recently been interested to find that the method of quadric inversion was distinctly suggested, though never developed, by Prof. Bellavitis of Padua no less than twenty-seven years ago. Considering the date of its appearance, the memoir, in the last paragraph of which this suggestion was made, is in many respects a remarkable one. It is entitled *Saggio di Geometria Derivata,* and will be found in the fourth volume of the *Nuovi Saggi dell' I. R. Accad. di Scienze, Lettere ed Arti di Padova.* Two years previously, that is to say in 1836, the same geometer had developed, very fully, the principles of the ordinary method of *cyclic* inversion; which latter, after a lapse of seven years, appears to have been first proposed in England by Mr. J. W. Stubbs, B.A., Fellow of Trinity College, Dublin, in a paper "On the Application of a new Method to the Geometry of Curves and Curve Surfaces," published in the Philosophical Magazine, vol. xxiii. p. 338.

Two curves are said to be inverse to each other, of course, when the several points of one are inverse to those of the other. The latter is sometimes referred to as the *primitive*, and the former as its *inverse*; the relation between the two curves, however, is a mutual one, and the distinction is merely introduced for convenience. In order to obtain clear conceptions of the various relations which exist between inverse curves, it will be found convenient to place the origin A outside the fundamental conic (F). The modifications to be introduced when the origin is placed elsewhere are quite obvious, and, except in a few instances, need not be specially alluded to.



3. Adopting terms introduced by Magnus, the origin A and the two points of contact B and C of the tangents from A to the fundamental conic (F) are called the *principal points*; the triangle, of which they form the corners, is termed the *principal triangle*, and its sides BC, CA, AB, respectively *polar* to A, C, B, are the *principal lines*. Occasionally it will be convenient to refer to B and C as distinct from A, which is always real; the two former will then be called the *fundamental* points, and the principal lines AB, AC, which there touch (F), will in like manner be called the *fundamental* lines *polar* to B and C.

4. This premised, it is manifest from Art. 2 that, in general, a point p has but one inverse point p'. The only exceptions, in fact, are the three principal points, each of which is obviously inverse to every point in the principal line which constitutes its polar. It is further evident that each point of the fundamental conic (F) coincides with its own inverse, and that the several points of (F) are the only ones in the plane which possess this property.

Hence may be inferred, without difficulty, the following theorem:

i. *If any two curves have* r-*pointic contact with each other at a point* p, not on a principal line, *their inverse curves will also have* r-*pointic contact with each other at the inverse point* p'.

Relative orders of inverse Curves.

5. The order of the curve inverse to a given curve (P) of the order n

may readily be ascertained. For in the first place, since (P) has n points on each principal line, its inverse must pass n times through each principal point (Art. 4); and secondly, since a line R drawn through the origin A intersects (P) in n points, none of which are, in general, situated on the principal line BC, polar to A, the same line R will intersect the inverse curve, not only in the n points coincident with the origin, but also in n points distinct therefrom. Hence

The complete *quadric inverse of any curve of the order* n *is a curve of the order* 2n, *which has multiple points, of the order* n, *at each of the three principal points.*

6. The term *complete* is here used because, under certain circumstances, the inverse curve will break up into one or more of the principal lines, each taken once or oftener, and a *proper* inverse curve (P′) of lower order, and which passes less frequently through the principal points. This, by Art. 4, will be the case whenever the primitive curve (P) passes through a principal point; and it is obvious that the order of (P′) will then be less than $2n$ by the total number of such passages. Further, the multiplicity of any principal point on (P′) will be less than n by the number of times the two principal lines, which there intersect, enter into the complete inverse,—in other words by the number of passages of the primitive curve (P) through the poles of those principal lines.

Hence, if a, b, c denote, respectively, the multiplicities of the principal points A, B, C on the curve (P), and if a', b', c' have the same significations relative to the inverse curve (P′) of the order n', we shall have

$$\begin{aligned} n' &= 2n-a-b-c, \\ a' &= n-b-c, \\ b' &= n-a-b, \\ c' &= n-a-c. \end{aligned}$$

These equations, by transformation, may readily be made identical with those which result therefrom by simply interchanging the accented and like non-accented letters; this shows, of course, that between proper inverse curves the same *mutual* relation exists as between inverse points. It is in virtue of this mutual relation that the theorems to be given hereafter are all conversely true; the enunciations of the converse theorems may therefore, in all cases, be omitted.

7. The above equations also furnish the following relations:

$$\begin{gathered} n'-(a'+b'+c')=(a+b+c)-n=n-n', \\ n'-a'=n-a, \quad n'-b'=n-c, \quad n'-c'=n-b, \end{gathered}$$

from which we learn that

i. *The difference between the orders of two inverse curves is numerically the same as that between the order of either, and the total number of its passages through the three principal points.* This latter difference, however, has opposite signs for the two curves.

ii. *For a curve to be of the same order as its inverse, and to pass the*

same number of times through each of the three principal points, it is necessary and sufficient, first, *that its order be equal to the total number of its passages through the three principal points*; *and* secondly, *that it pass as frequently through one fundamental point as through the other.*

It would be easy, by the second theorem, to determine the number and nature of the several curves of a given order which have inverse curves of the same order and like properties, relative to the principal points. This determination, however, as well as the solution of the allied question—

> *Under what conditions will a curve of given order coincide with its own inverse?*

will more appropriately form the subject of a separate paper. It will be sufficient to note here that a right line through the origin, and the fundamental conic itself, regarded as a primitive curve, are the simplest instances of the kind under consideration. Another example is also alluded to in Art. 11. Ex. 3.

Conics inverse to Right Lines.

8. From Arts. 2 and 6, as well as directly from elementary geometrical principles *, it follows that

i. *The inverse of a right line is, in general, a conic passing through the three principal points and the two intersections of the fundamental conic and the primitive line, as well as through the pole of the latter, relative to the former.*

It is only when the primitive line passes through a principal point that the inverse conic breaks up into the fixed principal line, polar to that point, and another right line (the proper inverse) through the principal point *opposite* to that fixed principal line (Art. 4). Thus:

ii. *The proper inverse of a right line passing through one of the two fundamental points is a right line passing through the other, and these inverse right lines always intersect on the fundamental conic.*

The following is an immediate consequence of this and the theorem i. of Art. 4:

iii. *If one of two inverse curves have* r*-pointic contact,* not on a principal line, *with a right line passing through a fundamental point, the other will have* r*-pointic contact with the inverse line, through the other fundamental point, and the points of contact, being inverse to each other, will be collinear with the origin.*

The modification which this theorem suffers when one of the points of contact is on a principal line, will shortly be fully considered.

9. The line at infinity has also its inverse conic (I), which is of importance in many inquiries connected with inversion. On observing (Art. 2) that the inverse of the infinitely distant point of any line R through

* An elegant demonstration of this theorem, identical with the elementary one alluded to, has been inserted by M. Chasles in Art. 209 of his excellent *Traité des Sections Coniques*, the first part of which has just reached me.

the origin is the middle point of the segment which (F) intercepts on R, and that (Art. 4) the infinitely distant points of (F) are necessarily also on (I), it will be seen that

i. *The conic* (I), *circumscribed to the principal triangle, which is inverse to the line at infinity, is similar and similarly situated to the fundamental conic, of which latter, in fact, it bisects all chords that converge to the origin.*

By means of this conic (I) the asymptotes to any inverse curve may be readily constructed. Since the next article, however, will be devoted to the construction of the tangent at any point whatever of an inverse curve, it will be sufficient here to note the following obvious corollaries of the above theorem:

ii. *The asymptotes of either of two inverse curves are respectively parallel to the right lines connecting the origin with the intersections of the other curve and the conic* (I), *which circumscribes the principal triangle and is, at the same time, similar and similarly situated to the fundamental conic* (F).

iii. *The conic inverse to a given right line will be a hyperbola, a parabola, or an ellipse, according as that line cuts, touches, or does not meet the conic* (I), *which circumscribes the primitive triangle and is similar and similarly placed to the fundamental conic* (F).

The conic (F) and the circle circumscribed to the principal triangle ABC have, of course, conjugate to BC, a second common chord H, inverse to that circle (Art. 8, i.), and this chord is clearly the only line in the plane whose inverse is a circle. The imaginary intersections of H with (I) are *inverse to the circular points at infinity,* and consequently lie, with the latter, on a pair of imaginary lines intersecting in the origin A.

Again, it is well known that all chords of (I) which subtend a right angle at A pass through a fixed point h *. The conics inverse to such chords are readily seen to be equilateral hyperbolas, and like their primitive lines they all pass through a fixed point—in fact, through the point h' inverse to h; this point h', moreover, is well known to be the intersection of the three perpendiculars of the triangle ABC, about which all the equilateral hyperbolas are circumscribed †. It further follows from a known theorem, that the chords of (I) which subtend a constant angle at A envelope a conic which has double contact with (I) at the inverse circular points, or, in other words, at the imaginary intersections of (I) and H; and conversely that all tangents to such a conic intercept on (I) an arc which subtends at A a constant angle ‡. The conics inverse to such tangents are, when the latter actually cut (I), hyperbolas whose asymptotes are inclined to each other at a constant angle—that is to say (in order to embrace all cases), *similar*

* Salmon's 'Conic Sections,' 4th ed., Art. 181, Ex. 2.

† *Ibid.* Art. 228, Ex. 1. Two distinct theorems in conics are thus brought, by inversion, into juxtaposition, and we have a simple example of the *duality* which this method, like that of reciprocal polars, imparts to every theorem.

‡ Chasles's *Sections Coniques*, Arts. 473, 474; also Salmon's 'Conic Sections,' Arts. 276, 277, 296.

conics. Now, by Arts. 4 and 6, the inverse of a conic which has double contact with (I) at the inverse circular points is, in general, *a quartic curve having double points at* A, B, C, *and touching, at the circular points, the line at infinity.* This curve, therefore, is also the *envelope of similar conics circumscribed to the triangle* ABC *. The point h clearly belongs to the above series of conics having double contact with (I), and H must be its polar relative to (I) †; so that we may resume as follows:

iv. *The right lines whose inverse conics are equilateral hyperbolas, all pass through the point* h *which is inverse to the intersection of the three perpendiculars of the principal triangle; the circle circumscribed to this triangle is the inverse of the polar* H *of the point* h, *relative to the conic* (I) *which is inverse to the line at infinity; the imaginary intersections of* H *and* (I) *are the* inverse *circular points, and all lines which envelope a conic* (Σ) *having double contact at these points with the conic* (I) *are inverse to conics which are similar to each other.*

Tangents to inverse curves at inverse points.

10. To a pencil of right lines L, whose centre is p, corresponds, by quadric inversion, a pencil of conics (L′) passing through the three principal points (Art. 8, i.) and the inverse point p'. To each element of the one pencil corresponds manifestly but one element of the other; so that the lines L through p, and the tangents L′, at p', to their respective inverse conics (L′) constitute two homographic pencils ‡; and since two corresponding rays of the latter coincide with pp', every other pair must intersect on a fixed line D (see figure). Since, moreover, to the rays p B, p C correspond, respectively, the rays p' C, p' B (Art. 8, ii.), it is obvious that the line D is simply one of the three diagonals of the complete quadrilateral p B p'C, the other two diagonals being pp' and BC. Hence if a be the intersection of the latter, the former D will, in virtue of a well-known property of the quadrilateral, pass through the harmonic conjugates d' and α of a, relative respectively to pp' and BC. Now α is at once recognized to be the pole, relative to the fundamental conic, of pp' or R; so that d, the inverse

* From this a new definition may be readily deduced of the interesting curve, of the fourth order and third class (with three cusps, and the circular points for points of contact of an infinitely distant double tangent), which presented itself to Steiner as the envelope of the line passing through the feet of the perpendiculars let fall from any point of a circle upon the sides of an inscribed triangle, and of which he has enunciated (merely) so many remarkable properties in a paper published in vol. liii. of Crelle's *Journal.* The curve is, in fact, *the envelope of an hyperbola circumscribed to an equilateral triangle, and having its asymptotes inclined to each other as are any two sides of that triangle.* The curve may also be generated as a hypocycloid, and appears to be identical with the one whose equation is given at p. 214 of Dr. Salmon's 'Higher Plane Curves.'

† Chasles's *Sections Coniques*, Art. 474.

‡ Chasles, "*Principe de correspondance entre deux objets variables,*" &c., *Comptes Rendus*, Dec. 24, 1855, and *Sections Coniques*, Art. 325. See also Cremona's '*Teoria geometrica delle curve piane,*' p. 7.

of d', must be the pole of D or $\alpha d'$, as well as the harmonic conjugate of A relative to pp' (Art. 2, i.). Consequently, from the fact that when L touches, at p, a primitive curve (P), the conic (L′), and hence its tangent L′, must touch, at p', the inverse curve (P′) (Art. 4, i.), we at once deduce the following theorem, by means of which the tangent at any point of a curve inverse to a given one may, for all positions of the origin, be readily constructed:

i. *The tangents, at two inverse points, to two inverse curves intersect on the polar, relative to the fundamental conic, of the harmonic conjugate of the origin with respect to their points of contact.*

Hence we may also deduce the following property:

ii. *To a multiple point on one of two inverse curves,* but not on a principal line, *corresponds, on the other, a multiple point of the same order of multiplicity, and the tangents to corresponding (inverse) branches all intersect on the polar, relative to the fundamental conic, of the harmonic conjugate of the origin, relative to the two multiple points.*

The reality, taction, and general distribution of the several branches will be the same at two such multiple points; the latter, in fact, will merely differ in certain secondary properties. For instance, a branch inflected at one of these points would not, in general, correspond to an inflected branch at the other; the latter branch, however, would have three-pointic contact, at this multiple point, with the conic inverse to the tangent of the inflected branch at the first multiple point.

11. The tangents at the principal points to two inverse curves may be thus investigated.

Exclusive of the principal points B and C, let the primitive curve (P) intersect the principal line BC in the points α, α_1, α_2, &c. . ., and conceive a right line R to rotate around the origin A. Exclusive of A this line R will intersect (P) in $n-a$ points p, respectively inverse to the $n'-a'$ points p' in which it intersects the inverse curve (P′) (Art. 7). It is clear, however, that whenever, by the rotation of R, p approaches one of the points α, p' will approach to coincidence with A, so that R will there touch a branch of (P′); and more generally, that if R should have $(r-1)$-pointic contact at α with (P) it would, at the same time, have r-pointic contact at A with one of the branches of (P′).

Similarly, if (P) intersect the fundamental line AC in the points β, β_1, β_2, &c. ., A and C being excluded, and a right line R_1, turning around B, intersect (P) in $n-b$ points p, their $n'-c'$ inverse points p' (Art. 7) will be the intersections of (P′) and the line R_2, inverse to R_1 and passing through C (Art. 8, ii.). Each pair of points p, p', moreover, will be collinear with A (Art. 2). Hence it follows that whenever, by the rotation of R, two points p and β approach each other, the inverse point p' will approach C, so that the line R_2 will there touch a branch of (P); and, as before, if R_1 have $(r-1)$-pointic contact with (P) at a point β, R_2 will have r-pointic contact at C with a branch of (P′).

In a similar manner the line R_2, which connects C and one of the intersections γ of AB and (P), has for its inverse a line R_1 touching, at B, a branch of (P′). All these cases are included in the following theorems:—

i. *The tangents at a principal point to one of two inverse curves are respectively inverse to the right lines which connect the intersections of the other curve and the principal line polar to that point, with the opposite principal point.*

ii. *If a non-principal line have* r*-pointic contact at a principal point with any branch of one of two inverse curves, the inverse line will have* (r−1)*-pointic contact with the other curve on the principal line polar to that principal point.*

The second of these theorems, as will be shown in the next article, is slightly modified when the line of r-pointic contact is a principal one; the first theorem, though still true, becomes susceptible of the following simpler enunciation:

iii. *If a branch of one of two inverse curves touch a principal line at a principal point, the other curve will have a branch touched by the polar of this point at the pole of that line.*

The following examples will serve as illustrations of these three theorems:

Ex. 1. The primitive being a right line intersecting the principal lines in α, β, γ, respectively (see figure), Aα will be the tangent at A to its inverse conic; and if Bβ, Cγ intersect in p, the inverse point p' will be the pole of BC relative to the inverse conic. This pole is always real; it may, moreover, be easily constructed, even when B and C are imaginary, on observing that p is also the intersection of the polar of α, relative to the fundamental conic, with the harmonic conjugate of αA, relative to BC and the primitive line.

Ex. 2. The primitive being a conic passing through the origin and intersecting the fundamental lines in β and γ, its inverse will be a cubic passing through B and C, and having a double point at A (Art. 6). This latter point will be a node, if the primitive conic cut BC in real points α_1, α_2; and Aα_1, Aα_2 will be the tangents thereat. It will be a conjugate or isolated point, however, when the primitive conic does not actually cut BC. The tangents to the cubic at B and C will, as before, intersect in the point p', inverse to the intersection p of Bβ, Cγ. If the primitive conic touch the latter lines in β and γ, in which case it is manifest that it cannot cut BC, then the cubic will have *real points of inflection at* B *and* C, *and, necessarily, a conjugate point at* A*. The line Ap is obviously the polar, relative to the primitive, as well as to the fundamental conic, of the intersection α of the lines BC, $\beta\gamma$; hence αA touches the primitive conic at A, and, by i., α is a point on the cubic; it is, in fact, the third point of inflection on this curve.

Ex. 3. The primitive being a conic touching the fundamental lines in

* The truth of a well-known theorem in cubics, 'Higher Plane Curves' (Art. 183), is here rendered visible.

the fundamental points, the inverse curve will be another conic possessing the same properties (Art. 6). The fundamental conic is not the only one of such a series which coincides with its own inverse (Art. 7); for there is obviously a second one which cuts every line through the origin in a pair of inverse points.

Singularities of inverse Curves.

12. If two of the intersections α, α_1, α_2 &c. .. of the primitive curve (P) with the principal line BC coincide; in other words, if (P) touch BC at α; then two of the branches of (P′) will unite to form a cusp at A, at which Aα will be the tangent. Similarly, if (P) touch a fundamental line, say AC at β, then on one of the branches of (P′) there will be a cusp at C, at which the line inverse to Bβ will be the tangent. In short—

i. *If one of two inverse curves touch a principal line at a non-principal point, the inverse of the connector of the point of contact with the opposite principal point will touch, at the pole of that line, a cusped branch of the other curve.*

The more general theorem is obviously this:

ii. *If one of two inverse curves have* r*-pointic contact with a principal line at a non-principal point,* r *branches of the other curve will coalesce so as to form a branch on which there will be a multiple point of the* r*th order at the pole of that line; and the sole tangent to this branch will be the inverse of the connector of the point of contact of the first curve with the opposite principal point.*

It may be added that, in general, the singularity at the principal point on this branch will be invisible or cusp-like, according as r is odd or even. Thus:

Ex. 1. If the primitive conic considered in Art. 11, Ex. 2, not only pass through the origin, but touch the principal line BC in α, its inverse cubic will, besides passing through B and C, have a cusp at A, the tangent at which will be Aα.

Ex. 2. If the primitive curve be a cubic with a node; then, the latter being taken as origin, the tangents thereat as fundamental lines, and the real stationary tangent of the cubic as the third principal line; the inverse curve will be a quartic which touches the latter line at the fundamental points B and C (Arts. 6 and 11, iii.), and has moreover a triple point at A, at which the sole tangent passes through the point of inflection on the primitive cubic. This tangent meets the quartic in four points coincident with A (𝔄. 𝔊. p. 190)*.

13. If the r-pointic contact in the preceding theorem occur at a principal point, we may conceive it to have arisen from the approach thereto of a point on the principal line, where the contact was $(r-1)$-pointic; the unique tangent to the branch of the inverse curve upon which there is a multiple point of the order $(r-1)$ will also, by this approach, have become coincident with a principal line †. Hence—

* See also Salmon's 'Higher Plane Curves,' Art. 217.

† The case corresponding to $r=2$ has already been considered in Art. 11, iii.

i. *If a branch of one of two inverse curves have* r*-pointic contact with a principal line at a principal point, the other curve will have, at the pole of that line, a multiple point of the order* $r-1$ *on a branch the sole tangent to which is the polar of that principal point.*

Ex. The primitive curve being a cubic which has the fundamental points B, C for points of inflection, and the fundamental lines for stationary tangents, and consequently another point of inflection α on BC, the inverse curve will be a quartic touching Aα at the origin, and passing twice through each of the fundamental points (Arts. 6 and 10). From the present theorem we conclude, further, that the latter points will be cusps on the quartic, and that the fundamental lines will be the tangents thereat (𝔄. 𝔈. p. 192, ix.).

14. From preceding articles the following properties may also be deduced:

i. *If one of two inverse curves have a multiple point on a principal line,* but not at a principal point, *the other will, in general, have a corresponding number of branches touching each other at the pole of that line*; *and at this pole the common tangent to these branches will be inverse to the connector of the first multiple point and the principal point opposite to the principal line on which it is situated.*

To obtain a clear conception of the modifications which may present themselves, it will suffice to consider the case where *the primitive curve* (P) *has a double point at* α *on the polar* BC *of the origin* A.

(a) The inverse curve will, in general, have a *tacnode* at A (𝔄. 𝔈. p. 164, figs. 17, 18)—in other words, two branches which there touch each other, the common tangent being Aα; these branches will, moreover, have three-pointic contact, at A, with the conics inverse to the two tangents at α to the primitive curve*. If one of the tangents at α coincide with αA, one of the branches of the inverse curve will be inflected at A (𝔄. 𝔈. fig. 21) (Art. 11, iv.). If one of the branches at α touch the primitive line BC, then one of the branches at A will be cusped, Aα being still the common tangent to the cusped and to the ordinary branch (𝔄. 𝔈. fig. 28). If both these singularities occur on the primitive at the same time, the inverse curve will present at A a triple point, with a single tangent, formed by an inflected and a cusped branch (𝔄. 𝔈. fig. 30).

(b) If the tangents at α coincide, so that the branches of the primitive curve there form an ordinary or *ceratoid* cusp (𝔄. 𝔈. fig. 16, *Spitze erster Art*), the conics of three-pointic contact at A with the corresponding branches of the inverse curve will also coincide, and the latter will possess a *ramphoid cusp* † (𝔄. 𝔈. fig. 19, *Spitze zweiter Art*), at which Aα will still be the sole tangent, meeting the cusp in *four* coincident points. In the special case where the tangent at the cusp α passes through the origin, the

* By Art. 11, i., the conics inverse to any right line whatever through α will have two-pointic contact, at A, with each branch of the tacnode.

† Prof. Cayley's term *cusp-node* is more appropriate (Quart. Journ. vol. vi. p. 74); the singularity in question may also be regarded as a stationary point on a stationary tangent, for the curve lies entirely on one side of the latter.

cusp at A on the inverse curve will also assume the ceratoid form, but it will be of higher order than the primitive one, since the tangent at A will meet it in *five*, instead of in *three* coincident points (𝔄. 𝔈. p. 167, iv.). If the principal line BC be itself the tangent at the ordinary cusp α, the inverse curve will have a triple point at A, the sole tangent $A\alpha$ at which will meet the curve in *five* coincident points. To the eye, this singularity will have the form of a point of inflection (𝔄. 𝔈. p. 174, fig. 29).

The following examples will illustrate the production, by inversion, of cusps of both kinds.

Ex. 1. The primitive curve being a cusped cubic, and the origin A being placed at its real point of inflection, let the stationary tangent be chosen as a fundamental line, and any line whatever through the cusp α as the polar of the origin. If the points B, C, in which the latter intersects the cubic and the stationary tangent, be considered as the fundamental points, the inverse curve will (Art. 6) be a quartic curve passing once through B, and twice through each of the points C and A. The tangent to the quartic at B will be the inverse of the line joining C to the third intersection γ of the cubic with the fundamental line AB (Art. 11, i.). The point C will be a *ceratoid cusp* on the quartic with CB for its tangent (Art. 13, i.). Lastly, $A\alpha$ will be the tangent at A to a *ramphoid cusp* on the inverse curve (b). The latter, therefore, is identical with the very remarkable quartic curve to which Prof. Sylvester's recent researches on the roots of equations of the fifth degree has imparted so great an interest (Phil. Trans. 1865). It is termed by him the 'Bicorn,' and is the one which, in Plücker's classification, is numbered xvi. (𝔄. 𝔈. p. 193). Since the primitive cubic from which it has been derived may itself be regarded as the inverse of a conic (Art. 12, Ex. 1), it is obvious that many properties of the Bicorn may be deduced from those of a conic, by double inversion, relative to two sets of principal points.

Ex. 2. The primitive being a conic, its inverse will, in general, be a quartic curve passing twice through each principal point (Art. 6).

All the ten varieties of such quartics which have been described by Plücker (𝔄. 𝔈. p. 195) correspond, in a very simple manner, to the different positions which the primitive conic may have*. Now it is well known† that, in general, two triangles may be inscribed in this conic, each of which shall, at the same time, be circumscribed to the principal triangle; whence we infer that two triangles may be inscribed in the quartic, so that a double point shall lie on each side of each triangle. A second inversion, therefore,

* For instance, if the principal triangle be self-conjugate relative to the primitive conic, the inverse quartic will have, at each principal point, *both its branches inflected* (Art. 11, ii.). In this case it is further obvious that two principal points must necessarily lie *outside* the primitive conic; so that *one* principal point will necessarily be a conjugate point on the quartic (Art. 11, Ex. 2). Inversion, in fact, renders visible the many curious properties, signalized by Plücker, which quartic curves present whenever their double points are also points of inflexion (𝔄. 𝔈. p. 199).

† Salmon's Conic Sections, 4th ed. p. 237; Chasles's *Sections Coniques*, Art. 246.

relative to either of these triangles will transform the quartic to a quintic (Art. 6) with three tacnodes, the varieties of which will correspond to those of the quartic. If, for instance, the primitive conic were inscribed in the original principal triangle, then the quartic would have three *ceratoid cusps*, and the quintic would be the remarkable one which Plücker has signalized (𝔄. 𝔈. p. 222, fig. 35) as possessing three *ramphoid cusps*.

(c) With respect to singularities of a higher order on the primitive curve, and on a principal line, little more need be added. To a tacnode at α would correspond, on the inverse curve, two branches touching the line Aα, and having three-pointic contact with each other at A (𝔄. 𝔈. p. 165, fig. 20). In fact, as a general rule, the contact of the branches at A is one higher in order than that of the corresponding branches at α. If α were a ramphoid cusp on the primitive, A would also be a ramphoid cusp, of higher order, on the inverse curve (𝔄. 𝔈. p. 170), and so on. It is worth observing, lastly, that although, by (a), the inverse of a tacnode at A is an ordinary node at α, on the polar of A, the latter will itself become a tacnode when approaches to coincidence with B or C (Art. 11, iii.), and similarly—

i. *If one of two inverse curves have a ramphoid cusp at a principal point, to which a principal line is the tangent, the other will also have a ramphoid cusp at the pole of that line, the tangent to which will be the polar of that point.*

This is manifest, in fact, from (b), on considering, with Professor Cayley *, a ramphoid cusp at B, with tangent BC, to arise from the coincidence of a ceratoid cusp at α with a node at B.

Special cases of quadric inversion.

15. The special cases of inversion which correspond to particular hypotheses relative to the position of the origin and to the nature of the fundamental conic are very numerous. The choice of these elements will depend of course upon the nature of the properties which are to be investigated by the method of inversion. A few only of the more useful of such special cases can be here noticed.

(1) *When the fundamental conic* (F) *is an hyperbola with its centre at the origin* A, its asymptotes constitute the fundamental lines, and their intersections with the line at infinity the fundamental points (Art. 3). Every right line parallel to one of these asymptotes has, for its inverse, a right line parallel to the other; and the two intersect on the fundamental hyperbola (Art. 8, ii.). The inverse of every other line in the plane is an hyperbola passing through the origin, and having its asymptotes parallel to those of the fundamental hyperbola (Art. 8, i.); moreover the conic (I) inverse to the line at infinity resolves itself into these asymptotes (Art. 9, i.). Every hyperbola which does not pass through the origin, but has its asymptotes parallel to those of the fundamental one, has, for its inverse, an hyperbola

* Quarterly Journal of Mathematics, vol. vi. p. 74.

possessing the same properties (Art. 6); and if the primitive have likewise its centre at the origin, the latter will also be the centre of its inverse (Art. 11, Ex. 3). The tangents to two inverse curves at two inverse points p, p' now intersect on a line D *bisecting* pp', and parallel to the diameter of the fundamental conic, which is conjugate to pp' (Art. 10).

(1*a*) *When the fundamental conic is an equilateral hyperbola, the origin being still at its centre,* the method of inversion becomes identical with the "hyperbolic transformation" investigated by Professor Schiaparelli, of Milan, in his interesting memoir, "*Sulla trasformazione geometrica delle figure*" *. The line D, upon which the tangents to two inverse curves at inverse points p, p' intersect, not only bisects pp', but is now inclined at the same angle as pp' to each of the fixed asymptotes of the fundamental hyperbola.

(2) *The fundamental conic being an ellipse and the origin at its centre,* the inverse of every right line in the plane will be an ellipse passing through the origin, and at the same time similar, as well as similarly placed, to the fundamental ellipse. The ellipse (I) inverse to the line at infinity now resolves itself to a *point* coincident with the origin. Every ellipse not passing through the origin, but similar and similarly placed to the fundamental one, has for its inverse an ellipse with the same properties; and should the primitive be likewise concentric with the fundamental ellipse, so also will be the inverse. The tangents at two inverse points p, p' to inverse curves again intersect on a line D which bisects pp', and is parallel to the diameter of the fundamental ellipse conjugate to pp'.

(2*a*) *When the fundamental conic is a circle with its centre at the origin,* we have, as already stated, the case of *cyclic* inversion, and the imaginary circular points at infinity are the fundamental points; the line D becomes, as is well known, the perpendicular to pp' through its middle point. From the theorems ii. and iii. of Art. 8 we should now infer that

(i) *The cyclic inverse of a right line through one of the circular points is a right line through the other, and the two intersect on the fundamental circle.*

(ii) *The cyclic inverse of a* simple *focus of any curve is always a focus of the inverse curve—unless the first focus should coincide with the origin, in which case the inverse curve would have cusps at the circular points at infinity* (Art. 12, i.).

It is important to notice that this theorem does not hold for a *double* focus f of the primitive curve—that is to say, for the intersection of the tangents to this curve at the circular points. For in this case the connectors of the inverse point f' with the circular points would merely *intersect* the inverse curve on the fundamental lines (Art. 11, i.); the line joining the points of intersection—a common chord of the *point-circle* f' and the inverse curve—would, however, bisect Af' perpendicularly. Thus it is that the cyclic inverse of the centre f of a primitive circle is not a

* *Memoria della Reale Accademia delle Scienze di Torino,* Serie II. tom. xxi.

centre of the inverse circle, but the inverse of the origin relative to the latter circle. If the primitive curve had points of inflection at the circular points, the inverse f' of the intersection f (a triple focus) of the stationary tangents thereat would not only be a focus of the inverse curve, but its corresponding directrix would bisect Af' perpendicularly. The focal relations of cyclic inverse curves, however, deserve closer examination; and I propose on another occasion to return to them.

(3) *When the fundamental conic consists of a pair of real, right lines* F_1, F_2 *intersecting at* F, the fundamental points B, C coincide with F, and the principal line BC with the harmonic conjugate of AF relative to F_1, F_2. The conic (I) inverse to the line at infinity is now an hyperbola, of which AF is a diameter, and the asymptotes of which are parallel to F_1 and F_2. Harmonic conjugates relative to the latter lines now constitute pairs of inverse lines, and the conic inverse to every other line, cutting BC say in α, is a conic touched at A and F by Aα and BC, so that the conics inverse to all lines parallel to BC are concentric, and have AF for a common diameter. The conics inverse to all lines passing through a fixed point a of AF have obviously three-pointic contact with each other at F, so that the conics inverse to lines parallel to AF have, at F, three-pointic contact with the hyperbola (I). All conics touching BC at F, but not passing through A, are inverse to conics having the same properties, and all conics passing through A and F, but not touching BC at the latter point, give by inversion conics of a similar description. The tangents at inverse points p, p' to two inverse curves now intersect on the harmonic conjugate of BC, relative to Fp, Fp'.

(3*a*) *The fundamental conic may consist of a pair of right lines, perpendicular to each other.* The results are then similar to, and somewhat simpler than those just noticed.

(4) *When the fundamental conic consists of a pair of imaginary right lines intersecting at a real point* F, the effects of inversion are analogous to those considered in (3). To secure real constructions, it is merely necessary to assume the *point-ellipse* F to be concentric with, similar, and similarly placed to a given ellipse (E). The lines FA and BC will then be conjugate diameters of (E), and any two inverse points p, p' will also lie on conjugate diameters. When the imaginary lines F_1, F_2 pass through the circular points, we have the following species of inversion:

(4*a*) *The fundamental conic consists of a point-circle* F. The principal line BC is now perpendicular to AF at F, and the connector of every pair of inverse points p, p' subtends a right angle at F. Inverse right lines through F, are perpendicular to each other, and the inverse of any right line in the plane is a conic, through A, to which AF is the normal at F. The circle (I) on AF as diameter, is the inverse of the line at infinity, and all right lines which touch one and the same circle, concentric with (I), are inverse to conics which are similar to each other—these conics being ellipses, parabolas, or hyperbolas, according as the circle in question is

greater than, equal to, or less than (I) (Art. 9, iii.). Lastly, every circle through F whose centre is on AF is inverse to a circle of the same kind as itself, as is also every circle passing through the points A and F.

(5) *When the origin* A *is on the fundamental conic* (F) the fundamental points B and C coincide with it, and the three principal lines coincide with the tangent at A. The conic (I) inverse to the line at infinity touches (F) at A and bisects all its chords through A, and the conic inverse to every other line in the plane has obviously three-pointic contact at A with (F). All lines which converge to one and the same point on the tangent at A are inverse to conics which have, at A, four-pointic contact with each other; hence right lines parallel to the tangent at A are inverse to conics having, at A, four-pointic contact with (I). Finally, the tangents to any two inverse curves, at inverse points p, p', intersect on a line D which touches (F) at the intersection of this conic and pp'.

A still more special case, into the details of which we cannot enter, is *when the fundamental curve is a parabola, and the origin at the infinitely distant extremity of its axis.* Inversion relative to a *conic and its focus* is also a special case which merits attention, but cannot be here considered.

Transformation correlative to quadric inversion.

16. Corresponding to the method of quadric inversion, there is of course a correlative method, which in certain inquiries is equally useful. It does not, however, require a separate exposition. It may also be remarked that the reciprocal polar, relative to the fundamental conic, of the inverse of any primitive curve, and the inverse of its reciprocal polar, lead at once to derivative curves, of which negative and positive Pedal Curves are special cases.

II. "On the Marsupial Pouches, Mammary Glands, and Mammary Fœtus of the *Echidna Hystrix*." By Professor Owen, F.R.S. Received February 18, 1865.

(Abstract.)

In a former communication on the generative economy of the Monotremata*, the author showed that the ovum left the ovarium with a spherical vitellus $1\frac{1}{2}$ line in diameter, and had attained a diameter of $3\frac{1}{2}$ lines in the uterus, the increase of size being due to increase of fluid between the chorion and vitelline tunics. This fluid, homologous with the albumen of the egg of oviparous vertebrates, did not coagulate in alcohol, and the only change presented by the vitellus of the largest observed ovum was a separation from the "food-yolk" of a "germ-yolk" in the form of a stratum of very minute granules, adhering to part of the membrana vitelli. There was no

* "On the Ova of the *Ornithorhynchus paradoxus*," Philosophical Transactions, vol. cxxiv. p. 555